Review of Introduction to Probability and Statistics

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Homework #3 Solutions

1. A 1-gallon can of paint covers an average of 513.3 square feet with a standard deviation of 31.5 square feet. Consider a sample of 40 of these 1-gallon cans. What is the probability that this sample will cover between 510 and 520 square feet per can?

Solution:

Since there is no reason to suspect that the single can paint coverage distribution is highly skewed, a 40-can sample will certainly allow the central limit theorem to kick in, so that the sampling distribution of the mean will be normal. Calculating the z-scores if the 510 and 520 ft^2 values,

$$z_1 = \frac{510-513.3}{31.5/\sqrt{40}} = -0.663, \quad z_2 = \frac{520-513.3}{31.5/\sqrt{40}} = 1.345$$

Using these values in a normal distribution calculator or table,

$\mathbb{P}(-0.663 < z < 1.345) = \mathbb{P}(z < 1.345) - \mathbb{P}(z < -0.663) = 0.9107 - 0.2536 = 0.657 \approx 0.66$

2. A manufacturing process produces nominally 800-lumen LED bulbs with a standard deviation of 12 lumens. To test the output of the manufacturing process, a sample of 140 bulbs are measured. If the mean of this sample is to be used for control purposes, what can be said with 99% probability about the maximum size of the error of the sample mean?

Solution:

With this sample size, we can assume that the sampling distribution of the mean will be normal. Thus, picking $z_{0.005} = 2.575$ for our margin of error calculation will provide the maximum error with 99% probability.

$$ME = z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 2.575\frac{12}{\sqrt{140}} = 2.6\ lumens$$

3. In six measurements of the melting point of tin, a chemist obtained a mean and standard deviation of 232.26 and 0.14 Celsius, respectively. What is the 98% confidence interval for the chemist's estimate of the actual melting point of tin?

Solution:

For this small sample, we can assume that the mean follows a Student's t distribution (which forces us to assume that the underlying probability distribution of the measurements is iid normal.) Using 5 degrees of freedom, the margin of error will be

$$ME = t_{\alpha/2}\frac{s}{\sqrt{n}} = 3.365\frac{0.14}{\sqrt{6}} = 0.19\ C$$

Thus, the 98% confidence interval is (232.07 C, 232.45 C).

4. A fuse manufacturer claims that at a certain current their fuse will blow in 12.4 minutes on average. To test this claim, 20 fuses were selected at random from a manufacturing lot and subjected to the specified test current. The mean time to blow for this sample was 10.63 minutes, with a standard deviation of 2.48 minutes. Does this data tend to support or refute the manufacturers claim? Assume that fuse time to blow follows a normal distribution.

Solution:

Since the sample is reasonably small, the sampling distribution of the sample mean will follow a Student's t distribution. We are not given a significance level in the problem, so we must pick our own. I'll pick $\alpha = 0.05$, but other choices could be reasonable as well.

Approach A: Construct a 95% confidence interval around the mean. With 19 degrees of freedom, we get from a Student's t calculator or table $t_{\alpha/2} = 1.729$. The margin of error is

$$ME = t_{\alpha/2}\frac{s}{\sqrt{n}} = 1.729\frac{2.48}{\sqrt{20}} = 0.959\ min$$

Thus our 95% confidence interval for our sample mean is (9.67 min, 11.59 min). Since this confidence interval does not capture the manufacture's value of 12.4 min, this sample tends to refute the manufacture's claim.

Approach B: Let's perform an hypothesis test. The null hypothesis will be that the population mean is equal to 12.4. The alternate hypothesis will be that the population mean is not equal to this value. The t-score for our sample mean is

$$t = \frac{10.63 - 12.4}{2.48/\sqrt{20}} = -3.19$$

There are 19 degrees of freedom, so that the p-value for this t-score will be 0.0054. (Note, you can get this number from Excel using the function "=T.DIST(3.19,19,false)"). Since the p-value to far below our specified significance level, we can reject the null hypothesis.

5. An infinite population is known to have a standard deviation that is 18% of the mean. When using a sample to measure the mean of this population, how big must the sample size be so that the standard error of the sample mean is 2% of the mean?

Solution:

We are given that $\sigma = 0.18\mu$ and we want $SE(\bar{x}) = 0.02\mu$. We also not that $SE(\bar{x}) = \sigma/\sqrt{n}$. Thus,

$$SE(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{0.18\mu}{\sqrt{n}} = 0.02\mu \quad \rightarrow \quad n = 81$$

6. An optical lens manufacturer purchases starting glass material in slabs and knows that historically the refractive index of the slabs has a variance of 1.26 X 10^{-4}. For a particularly critical product they sample the incoming glass and reject a shipment if the sample variance of a 20 piece sample exceeds 2.0 X 10^{-4}. Assuming that the sample is randomly drawn from a normal population, what is the probability that an historically typical shipment will be incorrectly rejected?

Solution:

The sample variance will follow a chi square distribution. Our chi square statistic will be

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{19(2.0 \times 10^{-4})}{1.26 \times 10^{-4}} = 30.2$$

Using Excel (the function call is "=CHISQ.DIST.RT(30.2,19)"), the right-tail probability is 0.049. Thus, there is a 4.9% probability of getting a variance of 2.0 X 10^{-4} or larger from a sample of 20 coming from a population with variance 1.26 X 10^{-4}.

7. An engineer wishes to investigate whether a process change will improve the yield of a manufacturing process. If y_1 is the yield of the existing process of record, and y_2 is the yield of the proposed process, write the most appropriate null and alternative hypotheses for an hypothesis test.

Solution:

H_0: $y_2 = y_1$. H_A: $y_2 > y_1$.

8. The specification for the breaking strength of a certain fishing line is 18 pounds. If five samples of that fishing line are obtained and tested to give a mean strength of 16.9

pounds, with a standard deviation of 0.9 lbs, use an hypothesis test to answer the question "is this line meeting its specifications?"

Solution:

The line will fail to meet its specification only if the strength of the line is less than 18 lbs. Thus, we need to perform a one-sided hypothesis test. I'll pick a significance level of $\alpha = 0.05$ (although another value could be chosen as well).

H_0: $\mu = 18$ lbs. H_A: $\mu < 18$ lbs.

Since the sample is small, the sampling distribution of the sample mean will follow a Student's t distribution (where we are forced to make the assumption that the underlying distribution is iid normal). With 4 degrees of freedom, $t_{0.05} = 2.132$ (note that we use t_α and not $t_{\alpha/2}$ since this is a one-sided test).

For our data, the t-statistic will be

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{16.9 - 18}{0.9/\sqrt{5}} = -2.73$$

With four degrees of freedom, the p-value is 0.026, which is less than our significance level of 0.05. Thus, there is a too small probability that this sample came about by pure chance if the null hypothesis is true. Thus, we can reject the null hypothesis in favor of the alternative hypothesis. (Note that if I was doing a two-tailed test, I would have to double this p-value. What would I conclude then?)

9. Two processes are being compared to determine if one produces wires with a lower resistance. For process 1, 32 samples are prepared yielding $\bar{x}_1 = 0.106\ \Omega$ and $s_1 = 0.008\ \Omega$. For process 2, 45 samples are prepared yielding $\bar{x}_2 = 0.093\ \Omega$ and $s_2 = 0.010\ \Omega$. At the 0.05 significance level, are these two processes different?

Solution:

H_0: $\mu_1 - \mu_2 = 0$ H_A: $\mu_1 - \mu_2 \neq 0$

This is a two-tailed test, and the sample sizes are large enough that the distributions will be about normal.

$$z = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}} = \frac{0.106 - 0.093}{\sqrt{\frac{0.008^2}{32} + \frac{0.010^2}{45}}} = 6.3$$

From this z-score, the two-sided p-value will be 2×10^{-10} (very small, indeed). We can safely reject the null hypothesis

10. The table below gives average weekly losses of worker-hours due to accidents at 10 warehouses before and after a certain safety program was put into place.

	Site 1	Site 2	Site 3	Site 4	Site 5	Site 6	Site 7	Site 8	Site 9	Site 10
Before	45	73	46	124	33	57	83	34	26	17
After	36	60	44	119	35	51	77	29	24	11

Using a 0.05 significance level, was the safety program effective?

Solution:

Since this data is before and after for the same sites, we want to use a matched pair sample. First, calculate the difference for each site. For these differences, the mean is 5.2 and the standard deviation is 4.1. Since an "effective" safety program is one that lowers the accident rate, we want to use a one-sided test. Because of the small sample, we will use a t-test.

H_0: $\mu_D = 0$ H_A: $\mu_D > 0$

$$t = \frac{\bar{D} - 0}{S/\sqrt{n}} = \frac{5.2 - 0}{4.1/\sqrt{10}} = 4.0$$

The p-value for the t-score, with 9 degrees of freedom, is 0.0015. Thus, we can reject the null hypothesis and conclude that the safety program is effective with 99.5% confidence.

11. An experiment makes 49 measurements and finds a mean of 12.4 and a standard deviation of 2.9. Create 95% confidence intervals for both of these statistics.

Solution:

For the mean, we can easily justify a normal sampling distribution by the central limit theorem. For a 95% confidence interval,

$$\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}\,,\ \bar{x} + 1.96\frac{s}{\sqrt{n}}\right) = \left(12.4 - 1.96\frac{2.9}{\sqrt{49}}\,,\ 12.4 + 1.96\frac{2.9}{\sqrt{49}}\right) = (11.6\,,\ 13.2)$$

For the standard deviation, we must assume that the underlying population of measurements is iid normally distributed. In this case, the variance will have a chi square distribution.

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

We can find the $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ values for 48 degrees of freedom from a table of from a software program (in Excel, use “=CHISQ.INV(0.025,48)”). This gives $\chi^2_{0.025} = 69.02$ and $\chi^2_{0.975} = 30.75$. Thus, our 95% confidence interval for the variance is

$$\left(\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right) = \left(\frac{48(2.9)^2}{69.02}, \frac{48s(2.9)^2}{30.75}\right) = (5.85, 13.13)$$

To get the confidence interval for the standard deviation, we can take the square root of the variance confidence interval (a pretty good approximation).

$$\left(\sqrt{5.85}, \sqrt{13.13}\right) = (2.4, 3.6)$$