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Online Review Course of
Undergraduate Probability and Statistics

Review Lecture 9
Continuous Random Variables

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Course Website: www.lithoguru.com/scientist/statistics/review.html

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What is a Random Variable?

- Random Variable: a real-valued function of the outcomes of the experiment
  - $\Omega \rightarrow \mathbb{R}$ (maps sample space onto the real numbers)
- Example
  - Ω = {all UT students}, X = height of randomly selected student
- Discrete versus Continuous random variable
  - Option 1: round height measurement to the nearest inch. Result = discrete RV
  - Option 2: measure height with infinite precision. Result = continuous RV.

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Continuous Random Variable

- Consider a discrete RV:

$$\mathbb{P}(a \le X \le b) = \sum_{all\ a \le x \le b} p_X(x)$$

- Now let the distance between values of x go to zero

PDF, probability density function

$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

Replace summation with integral
Replace PMF(x) with PDF(x) dx

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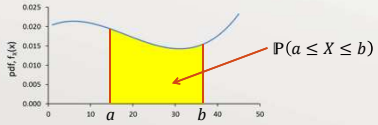
Probability Density Function

- Definition of PDF:

$$f_X(x) = \lim_{\delta \to 0} \frac{\mathbb{P}(x \le X \le x + \delta)}{\delta}$$

Probability per unit length along x

- Probability is the area under the pdf curve



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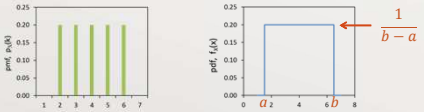
PDF Properties

- Non-negativity: $f_X(x) \ge 0$
- Normalization: $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- The probability at a point is zero: $\mathbb{P}(X = a) = 0$
- Example: Uniform probability



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Expectation and Variance

Discrete:

$$E[X] = \sum_{all\ x} x\, p_X(x) \qquad var[X] = \sum_{all\ x} (x - E[X])^2\, p_X(x)$$

Continuous:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx \qquad var[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx$$

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Uniform PDF

- Uniform pdf between a and b, zero outside this range

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & otherwise \end{cases}$$

$$E[X] = \int_a^b \frac{x}{b-a} dx = \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b+a}{2}$$

$$var[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx = \frac{(b-a)^2}{12}$$

Cumulative Distribution Function

- CDF for a continuous RV

$$F_X(b) = \mathbb{P}(X \le b) = \int_{-\infty}^{b} f_X(x) dx$$

- CDF is monotonically non-decreasing

$$\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a)$$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Normal Distribution

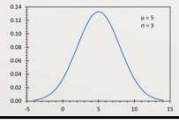
- Also called the Gaussian distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = N(\mu, \sigma^2)$$

$$F_X(x) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)\right]$$

$$E[X] = \mu$$

$$var[X] = \sigma^2$$



Normal Distribution

- Define $z = \frac{x-\mu}{\sigma}$ (z-score)

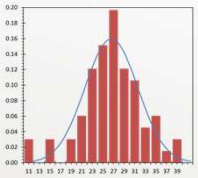
"standard normal"

$$If\ X = N(\mu, \sigma^2)\ \ then\ \ Z = N(0,1)$$

- A linear function of a normally distributed RV produces a normally distributed RV
  - Let $Y = \alpha X + \beta$.
  - If $X = N(\mu, \sigma^2)$ then $Y = N(\alpha\mu + \beta, (\alpha\sigma)^2)$

There are Many, Many PDFs

- Some common PDFs:
  - Exponential distribution
  - Cauchy distribution
  - Student's t-distribution
  - Chi-squared distribution
  - Log-normal distribution
  - Beta distribution
  - Gamma distribution
  - Etc.
- Wikipedia is a very good source for distributions and their properties



Histograms are used to compare to PDFs

Review #9: What have we learned?

- How are discrete and continuous RVs related?
- How are PDF and CDF related for a continuous RV?
- Know how to calculate expectation and variance given a PDF.
- What is the normal distribution and some of its important properties?

